Normal Curves of Genus 6, and their Groups of Birational Transformations.

By Virgil Snyder.

1. The canonical form to which an algebraic curve of given genus can be reduced is one of the fundamental problems in the theory of birational transformations. The simplest forms of curves of genus 3 and their corresponding groups have been found by Wiman,* who also made a study of space curves of genus 4, and an outline of that of curves of genus 5. The forms and properties of plane curves of genus 4 have been determined by Miss Van Benschoten;† the classification of those of genus 5 is well under way. The present paper has for its purpose the determination of the groups of birational transformations which leave curves of genus 6 invariant and to discuss various properties of such curves. This configuration is interesting from the fact that it is the lowest genus which can not be defined by the complete intersection of quadric spreads in hyperspace, and that only one of the defining spreads can be assumed at will.

The general curve of genus 6 can be reduced to a sextic c_6 with four double points. The only exceptions are the hyperelliptic curve and the non-singular quintic. When the curve is reduced to another of the same order by a non-linear transformation it must contain a linear g_6^2, of which the points of each group are not collinear. Since this is a special series, it can be determined by adjoint cubic curves ϕ_3. But the $\infty^2\,\phi_3$ having the four double points and any other three points of c_6 for basis points define not a g_6^2, but g_7^2; hence: *all transformations which transform a non-hyperelliptic curve of genus* 6 *and order* 6 *into itself or any other curve of the same order birationally can be expressed by collineations and quadric inversions.* Moreover, every transformation generated by these must, in this case, be either linear or quadratic.

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* *Bihang till Svenska Vet. Akad. Handlingar*, Band XXI (1895).

† A. L. Van Benschoten, *On the Transformations Which Leave the Algebraic Curves of Genus* 4 *Invariant*, Cornell dissertation, 1908.

The following cases are to be considered:

(a) The normal curve is a c_6 with four double points ($4P_2$) at the vertices of a quadrangle.

(b) The c_6 has three collinear double points, and one other one.

(c) The c_6 has a triple point P_3 and a double point.

(d) The curve has a g_5^2.

(e) The curve is hyperelliptic.

§ 1 (a). c_6 *with Four Double Points, General Case.*

2. This curve has $5\,g_4^1$, formed by the pencils of straight lines through each of the nodes, and the pencil of conics through all of them. When more than five g_4^1 exist, the curve has an infinite number of such series and can not be reduced to a sextic. These series must permute among themselves; hence, *curves of form* (a) *can have no group of order larger than* 120. Let the four double points be (1, 0, 0), (0, 1, 0), (0, 0, 1), (1, 1, 1). The $\infty^3\,\phi_3$ through these points will be of the form

$$a\,(x^2y-xyz)+b\,(xy^2-xyz)+c\,(x^2z-xyz)+d\,(y^2z-xyz)+e\,(xz^2-xyz)+f\,(yz^2-xyz)=0.$$

If we now put

$$\rho x_1=xy\,(x-z),\quad \rho x_2=xy\,(y-z),\quad \rho x_3=xz\,(x-y),$$
$$\rho x_4=yz\,(y-x),\quad \rho x_5=xz\,(z-y),\quad \rho x_6=yz\,(z-x),$$

then between the x_i exist the five following identities:

$$\left.\begin{array}{ll} x_2x_5-x_4x_5-x_2x_6=0, & x_3x_6+x_1x_5-x_1x_6=0,\\ x_1x_4+x_2x_3-x_3x_4=0, & x_1x_4-x_1x_6+x_2x_6=0,\\ x_4x_5+x_3x_6-x_3x_4=0. & \end{array}\right\}\quad(1)$$

The equation of any c_6 having the above points for double points can be expressed as a quadratic relation among the x_i. It contains 21 terms, or 20 constants, but five of these can be expressed in terms of the others by means of (1). Thus all the $15=3p-3$ constants appear in the one equation

$$\Sigma\,a_{ik}\,x_i\,x_k=0. \qquad (2)$$

Now consider x_i as homogeneous point coordinates in linear space of five dimensions R_5. The systems (1) and (2) define six quadric spreads which have a curve in common. Any ϕ_3 cuts c_6 in 10 points, hence an R_4 defined by $\Sigma\,b_i\,x_i=0$ will cut the normal curve in 10 points. If the $c_{10}^{(5)}$ be projected from any point upon it and the projecting cone cut by an R_4, the resulting $c_9^{(4)}$ can not have a

double point. In other words, $c_{10}^{(5)}$ can have no trisecants. By further projections this curve becomes $c_8^{(3)}$ and $c_7^{(2)}$ respectively. A c_7 with 9 double points can not be reduced to a c_6 by means of ϕ_4 passing through an arbitrary set of basis points, but if one simple basis point be assumed, three others can be found in five different ways, such that $\infty^2 \phi_4$ can be passed through them and the nine double points.*

If these curves be used for the transforming system, the transformed curve will be a c_6 with four double points. Since $c_{10}^{(5)}$ can be projected from certain points upon it into $c_6^{(2)}$, it is possible to find tetrads of points upon it such that through them can be passed $\infty^2 R_4$. It is necessary and sufficient that the four basis points lie in R_2. If $c_{10}^{(5)}$ be projected from one of them into $c_9^{(4)}$, the other three will go into points lying on a straight line; hence $c_9^{(4)}$ *has at least five trisecants.* If $c_9^{(4)}$ be projected into R_3 from one of the points of intersection with a trisecant, $c_8^{(3)}$ will have a double point. Finally, $c_8^{(3)}$ is projected from the double point into our plane c_6.

3. Let the systems (1) and (2) which define $c_{10}^{(5)}$ be denoted by $F_1, \ldots, F_6$. Among the spreads of the linear system $\Sigma \lambda_i F_i = 0$ are ∞^4 which can be expressed in terms of five variables. These particular spreads are the four-dimensional quadric cones, having a point for vertex. The associated values of λ_i are found by equating the discriminant of the system $\Delta(\lambda_1, \ldots, \lambda_6) = \Delta(\lambda)$ to zero. If λ_i be regarded as point coordinates in R_5, $\Delta(\lambda) = 0$ is a four-dimensional spread of order 6. The corresponding locus of the vertex of the cones is obtained by equating the determinant of $F_{ik} = \dfrac{\partial F_i}{\partial x_k}$ of order 6 to zero. It is also of order 6. Between this spread M and Δ exists a one-to-one point correspondence. If from a point of M the curve $c_{10}^{(5)}$ be projected into R_4, $c_{10}^{(4)}$ will lie on an $F_2^{(3)}$. But there are values of λ for which F can be expressed in terms of four variables. These spreads are four-dimensional quadric cones having a line for vertex. The values of λ are obtained by equating all the first minors of Δ to zero. The corresponding configuration on M is a ruled hypersurface S. If $c_{10}^{(5)}$ be projected into R_3 from a line of S which is a bisecant of the curve, the $c_8^{(3)}$ is of type (4, 4) on a quadric surface, and has three actual double points.†

In no case can $c_9^{(4)}$ have an actual double point, as it would give rise to $c_7^{(3)}$; but this curve has a g_3^1, hence belongs to those of type (c). Through every point

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* Clebsch, *Geometrie*, Vol. I, p. 695.

† See Riemann, in *Crelle*, Vol. LIV, § 13; Clebsch, *Geometrie*, Vol. I, p. 693, foot-note; Brill, *Math. Ann.*, Vol. I, p. 401, and Vol. II, p. 471.

of $c_{10}^{(5)}$ can be passed five R_2, each of which cuts the curve in three remaining points. If $c_{10}^{(5)}$ be projected from such an R_2 into R_2, the result is $c_6^{(2)}$ with four double points. Thus the g_6^2 in the $c_6^{(2)}$, and some fixed point on the curve, can never be a partial series g_7^2 contained in g_7^3, but two such points can be found so that the resulting g_8^2 is a partial series contained in the $g_8^{(3)}$.*

4. The four basis points project on $c_6^{(2)}$ into the four points in which any conic through the double points cuts the curve. One of them is arbitrary and the others are then fixed. Similarly, the two fixed points which are the images of the double point of $c_8^{(3)}$ are the residual points in which the line joining two nodes cuts the curve again. The adjoint ϕ_3 are made up of the straight line joining the other two nodes, and the ∞^3 conics through the first two; as subgroup we have the degraded conics formed by the line joining the second pair of nodes and an arbitrary line of the plane.

5. The curve $c_{10}^{(5)}$ is a double curve on M, the Jacobian of the system of quadrics. Among the lines S, some are bisecants, some simple secants, but in general they do not intersect $c_{10}^{(5)}$. If the curve be projected from a general line of S, its image in R_3 is a $c_{10}^{(3)}$ of type (5, 5) on a quadric. It has 10 actual double points. On the other hand, if $c_{10}^{(5)}$ be projected into R_3 from any bisecant, the resulting $c_8^{(3)}$ will have no double points. It is the partial intersection of a cubic and a quartic surface, the residual being a rational $c_4^{(3)}$.

Given any point P on $c_{10}^{(5)}$. Associated with it are five sets of three points each, P_1^k, P_2^k, P_3^k $(k=1,\ldots,5)$, such that each set and the point P lie in a plane. If these points be called a particular group, we may say: *The $c_8^{(3)}$ obtained by projecting $c_{10}^{(5)}$ into R_3 from a line joining any two points of a particular group will always have at least one double point.*

If $P_1^1=P_2^1$, then through the line PP_1^1 can be passed two R_2, each cutting $c_{10}^{(5)}$ in two other points. The $c_8^{(3)}$ obtained by projecting from such a line must have at least two double points; but since a g_3^1 on $c_{10}^{(5)}$ is excluded, if $c_8^{(3)}$ has two double points, it has three. Since the points associated with P_1^1 must be P, and the two remaining associates of P, we now have the following theorem:

The necessary and sufficient condition that a line joining two corresponding points of the same particular group on $c_{10}^{(5)}$ be the vertex of a quadric cone on which $c_{10}^{(5)}$ lies is that one point on $c_{10}^{(5)}$ is common to two particular groups belonging to the other.

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* This is a direct application of Noether's theorem of reduction. See Segre, "Introduzione alla Geometria sopra un Ente Algebrico Semplicemente Infinito," *Ann. di Mat.* (2), Vol. XXII (1894).

Thus, there can never be a simple coincidence of associated points on $c_{10}^{(5)}$. In each case the coincidences appear in sets of three.

6. Every non-hyperelliptic curve of genus p greater than 3 has one or more linear g_{p-1}^1. By the Riemann-Roch theorem the residual series is also a g_{p-1}^1. In the canonical curve in R_{p-1} these series must be cut by R_{p-2}, arranged in reciprocal sets. A series of R_{p-3} can be found having $p-1$ points on $c_{2p-2}^{(p-1)}$. Any R_{p-2} through these points will cut the curve in $p-1$ further points, which also lie on a R_{p-3}. The curve lies on a quadric spread in R_{p-1} which can therefore be projectively generated by the R_{p-2} of each series. This is possible only when the equation of one quadric on which $c_{2p-2}^{(p-1)}$ lies can be reduced to contain but four variables. If this hypercone be projected from its double R_{p-4} into R_3, $c_{2p-2}^{(p-1)}$ will project into $c_{2p-2}^{(3)}$ lying on a quadric surface.

The curve is of type $(p-1, p-1)$ and has $(p-1)(p-4)$ double points. *A quadratic identity between four adjoint curves can be found (by means of the equation of the curve itself) for every non-hyperelliptic curve of genus* $p > 3$.*

7. The linear transformations which leave $c_6^{(2)}$ invariant must also permute the double points among themselves. If

$$A_1 \equiv (1, 0, 0), \quad A_2 \equiv (0, 1, 0), \quad A_3 \equiv (0, 0, 1), \quad A_4 \equiv (1, 1, 1),$$

all the possible linear transformations are contained in the g_{24} formed by the 4! permutations of these points. As generating operations we may take the three harmonic homologies

$$(A_1 A_2)(A_3)(A_4) = (x_1 x_2)(x_3 x_4)(x_5 x_6),$$
$$(A_1 A_3)(A_2)(A_4) = (x_1 x_6)(x_2 x_4)(x_3 x_5),$$
$$(A_1 A_4)(A_2)(A_3) = (x_1 x_3)\begin{pmatrix} x^2 & x_4 & x_5 & x_6 \\ x_3 - x_1 + x_2 & -x_1 + x_2 - x_4 & -x_3 + x_1 + x_5 & x_5 - x_3 - x_6 \end{pmatrix}.$$

Since the quadratic identities (1) which are independent of the c_6 simply permute among themselves, in order to obtain the most general c_6 of $p = 6$ which is invariant under any group contained in the above octahedron group, simply write a general quadratic relation among the x_i and impose such conditions as will leave its form unaltered when operated upon by the generators of the group.

The only other operation which can leave the curve invariant is the quadratic

*This theorem does not contradict that stated by Noether, *Math. Ann.*, Vol. XVII, p. 441. There the basis points a_1, a_2, ... are chosen arbitrarily.

inversion, having any three of the double points for basis points. That determined by $A_1 A_2 A_3$ and leaving the point A_4 fixed can be expressed in the form

$$(x_1 x_6)(x_2 x_5)(x_3 x_4).$$

The pencil of conics passing through all four double points defines a fifth linear series g_4^1, and may be denoted by A_5. The operation of inversion as to $A_1 A_2 A_3$ changes the pencil of straight lines through A_4 into A_5, and conversely. Hence

$$(A_4 A_5)(A_1)(A_2)(A_3) = (x_1 x_6)(x_2 x_5)(x_3 x_4).$$

These four generators will define the symmetric group of order 120, and proper combinations of them will define any group contained within it.

The largest period of any birational transformation which leaves a c_6 *of type* (a) *invariant is six.* These operations and the corresponding equations can now be immediately written down. In particular, if the curve belongs to the group generated by $(A_1 A_2)$, $(A_1 A_3)$, $(A_4 A_5)$, its equation is of the form

$$A \sum_{i=1}^{6} x_i^2 + B(x_1 x_2 + x_1 x_3 + x_2 x_4 + x_3 x_5 + x_4 x_6 + x_5 x_6) + C(x_1 x_6 + x_2 x_5 + x_3 x_4) = 0,$$

when proper use is made of equations (1). If it be invariant under $(A_1 A_4)$ also, $A = -2$, $B = 2$, $C = -1$. Expressed in terms of x, y, z this equation defines the c_6 having the maximum group of order 120. Its form is

$$2\Sigma x^4 y z + 2\Sigma x^3 y^3 - 2\Sigma x^4 y^2 + \Sigma x^3 y^2 z - 6 x^2 y^2 z^2 = 0.$$

§ 2 (b). *Three Double Points Collinear.*

8. If this form be inverted as to a triangle of nodes, the third node on one of the sides of the triangle becomes a tacnode, hence the latter configuration is as general as the former. Let the tacnode be at (0, 0, 1), $x + a y = 0$ the equation of the tacnodal tangent. From the system of adjoint cubics we may write

$$\rho x_1 = x^2 y,\ \rho x_2 = x y^2,\ \rho x_3 = x^2 z,\ \rho x_4 = x z^2 + a y z^2,\ \rho x_5 = y^2 z,\ \rho x_6 = x y z,$$

from which the quadratic relations

$$x_1 x_6 - x_2 x_3 = 0,\quad x_1 x_5 - x_2 x_6 = 0,\quad x_3 x_5 - x_6^2 = 0,$$
$$x_2 x_4 - x_6(x_6 + a x_5) = 0,\quad x_1 x_4 - a x_6^2 + x_3 x_6 = 0$$

at once follow. Any curve c_6 having this configuration of nodes is completely defined by $\Sigma a_{ik} x_i x_k = 0$.

The only linear transformations that will leave this configuration invariant are of the form

$$\rho x_1^1 = x_1, \;\; \rho x_2^1 = x_2, \;\; \rho x_3^1 = k x_3, \;\; \rho x_4^1 = k^2 x_4, \;\; \rho x_5^1 = k x_5, \;\; \rho x_6^1 = k x_6,$$

or of the form

$$\rho x_1^1 = x_2, \;\; \rho x_2^1 = x_1, \;\; \rho x_3^1 = x_5, \;\; \rho x_4^1 = x_4, \;\; \rho x_5^1 = x_3, \;\; \rho x_6^1 = x_6.$$

In the first case $k^2 = 1$ or $k^4 = 1$, but the latter is possible only for $x_4^2 = x_1 x_2$, which is hyperelliptic, $p = 2$. For any odd value of k, c_6 is composite. By letting $a = 1$, which is no restriction, the curve having the non-cyclic G_4 becomes

$$a_1 x^3 y^3 + a_2 x^2 y^2 (x^2 + y^2) + a_3 z^4 (x + y)^2 + a_4 x^2 y^2 z^2 + a_5 (x^4 z^2 + y^4 z^2) + a_6 z^2 (x^2 + y^2)^2 = 0.$$

It was shown that no inversion can leave either type invariant, hence: *The most general birational group which leaves a c_6 of type* (b) *invariant is the non-cyclic linear G_4.*

If $a = 0$, the form may be written $\Sigma a_i x_i^2 = 0$.

The case of two tacnodes, and in particular, of one tacnodal tacnode passing through the other tacnode, can all have the non-cyclic G_4. The case of three consecutive collinear nodes and that of the oscnode are equivalent. The former is invariant under a harmonic homology, the latter under an inversion with coincident fundamental points. Moreover, both forms are invariant under one other harmonic homology, commutative with the first operation; hence these have also the same g_4.

Finally, if all four double points are consecutive, they must lie on a conic. If $(0, 0, 1)$ be the point, $y = 0$ the tangent and $zy = x^2$ the conic on which the nodes lie, the system of adjoint ϕ_3 may be written in the form

$$\rho x_1 = y z^2 + x^2 z, \;\; \rho x_2 = x y z + x^2, \;\; \rho x_3 = y^2 z, \;\; \rho x_4 = x^2 y, \;\; \rho x_5 = x y^2, \;\; \rho x_6 = y^3,$$

and the quadratic relations become

$$x_2 x_5 = x_3 x_4 + x_4^2, \quad x_2 x_6 = x_3 x_5 + x_4 x_5, \quad x_4 x_6 = x_5^2,$$
$$x_1 x_5 = x_2 x_3, \quad x_1 x_6 = x_3^2 + x_3 x_4, \quad \Sigma a_{ik} x_i x_k = 0.$$

The only operation which will leave these forms invariant is changing the signs of x_2 and x_5. It is a harmonic homology with center on the tangent and axis through the node.

9. It will be noticed that in all forms under (b) at least one of the quadratic identities involved but three of the variables. In each case $\Sigma a_{ik} x_i x_k = 0$ may be particularized to include many more such forms, even when the form of the nodes is prescribed.

Any quadratic relation involving but three adjoint ϕ may be written in the form

$$2\phi\phi'' - \phi'^2 = 0,$$

which, with the other quadratic relations, defines c_6 without extraneous factors; at every variable point in which ϕ, ϕ' intersect, the former touches c_6. Similarly for ϕ'', ϕ'. If the curve $a\phi + b\phi' + c\phi'' = 0$ cuts c_6 in the two sets $(\phi_1, \phi_1', \phi_1'')$, $(\phi_2, \phi_2', \phi_2'')$, then c_6 may also be written in the form

$$2(\phi\phi_1'' + \phi_1\phi'' - \phi'\phi_1')(\phi\phi_2'' + \phi_2\phi'' - \phi'\phi_2') = (a\phi + b\phi' + c\phi'')^2;$$

hence if there be two contact curves, there will be an infinite number. These systems are the images of the tangent R_4 to the cone $2\phi\phi'' - \phi'^2 = 0$ having an R_2 for vertex. Since through a set of $p - 1 = 5$ points of contact both ϕ and ϕ' pass, they are the basis points of a pencil. In general, if l be any line and ψ_{n-2} be a curve of order $n-2$ passing through the nodes, a special group of $p-1$ points and $n-1$ of the intersections of c_n, l, then the net

$$a l\phi + b l\phi' + c\psi = 0$$

will have $p - 1 + n - 3$ fixed basis points in addition to the nodes; this leaves $p + 2$ variable intersections. This net can now be used to transform c_n birationally into another of order $p + 2$. The point (0, 0, 1) is a triple point P_3 on c_{p+2}, since the pencil of straight lines through it corresponds to $a\phi + b\phi' = 0$. A contact curve must go into a contact curve; hence some line of the pencil $ax_1 + bx_2 = 0$, counted twice, is image of the contact curve. It is a factor of an adjoint curve; the remaining nodes lie on a curve of order $p - 3$. Now let a_{p-1} be a curve passing through the triple point and all the double points but one, b_{p-2} a curve passing through all the double points, $k \equiv x_1 + kx_2 = 0$ be a line passing through the triple point and the node not lying on a_{p-1}. The net formed by $x_1 b$, $x_2 b$, a can now be used to transform c_{p+2}. The transformed curve will be of order $p + 1$, and the pencil through (0, 0, 1) remains invariant; as before, one of its lines must count double. The remaining double points lie on a curve of order $p - 4$, but this is impossible unless the special line contains a second double point, hence (0, 0, 1) is a tacnode. For $p = 4$ and $p = 5$ the curve can not be reduced to a simpler form, but for $p > 5$, it is always possible to further reduce the order of the curve.

10. These steps can be easily interpreted geometrically. The ϕ represents an R_4, tangent to the quadric cone of R_5, having an R_2 for vertex. Each tangent R_4 touches $c_{10}^{(5)}$ in five points, the points of contact lying in an R_3. Let A, B, C, D, E be the points of contact. Project $c_{10}^{(5)}$ from A into an R_4 not passing

through A. The $c_9^{(4)}$ contains the images A', B', C', D', E'; it will be touched by an R_3 in B', C', D', E' which also passes simply through A'. The four points of contact lie in an R_2. Now project $c_9^{(4)}$ from B' into R_3 not passing through B'. An R_2 touches $c_8^{(3)}$ in C'', D'', E'' and passes through A'', B''. The points of contact are collinear. Project $c_8^{(3)}$ from C'' into R_2 not passing through C''. The $c_7^{(2)}$ will have a tacnode, and the images of the other points from which the successive curves were projected are the residual intersections of the tacnodal tangent and the curve. The g_7^2 formed by the lines of the plane of c_7 is such that if A'' be adjoined to each group, the series g_8^2 is incomplete, being contained in a g_8^3, similarly for g_9^4, g_{10}^5. If we construct a system of $\infty^5\, \phi_4$ such that when two further basis points (B''', E''') are given, a g_8^3 will be defined, and further such that the g_8^2 obtained by fixing one more basis point will have as partial series the straight lines of the plane, it is only possible when the seven remaining double points lie on a conic.*

11. If $c_{10}^{(5)}$ be projected into R_2 from the vertex R_2 of the quadric cone, the result is a conic, counted five times. If $c_{2p-2}^{(p-1)}$ be projected from an R_{p-4} vertex of a quadric cone on which the curve lies into R_3, the resulting conical curve will cut each generator in $p-1$ points and have $(p-1)(p-4)$ actual double points. If $c_{2p-2}^{(3)}$ be projected into R_2 from one of these double points, the $c_{2p-4}^{(2)}$ will have $p-3$ branches touching each other at a common point, and $(p-1)(p-4)-1$ other double points lying on ϕ_{p-3}. Both this form and the preceding one can be obtained without the use of special groups.

12. Now suppose there are two quadratic relations which involve but three variables. Through every point of $c_{10}^{(5)}$ now pass two tangent R_4, each of which touches $c_{10}^{(5)}$ in four other points. In the two correspondences formed by the tangent R_4, it will happen for a finite number of points that the two R_4 will also have another point of $c_{10}^{(5)}$ in common. Now proceed as before, first projecting from one of these points, then from the other. The $c_8^{(3)}$ has an actual double point, through which pass two planes, each of which touches $c_8^{(3)}$ in three other points, the points of contact being collinear.

§ 3 (c). c_6 *has a* g_3^1.

13. When a curve of genus 6 and having a g_3^1 is reduced to c_6, the curve must have a triple point.†

* This same result was obtained by Kraus, *Math. Ann.*, Vol. XVI, by a partly different method.

† Amodeo, "Curve k-gonali," *Ann. di Mat.* (2), Vol. XXI (1893), p. 221.

If the triple point be chosen at (0, 0, 1) and the double point at (0, 1, 0), the system of adjoint ϕ_3 may be written

$$\rho x_1 = x^2 z,\quad \rho x_2 = xyz,\quad \rho x_3 = y^2 z,\quad \rho x_4 = x^3,\quad \rho x_5 = x^2 y,\quad \rho x_6 = xy^2,$$

from which

$$\frac{x_1}{x_2} = \frac{x_2}{x_3} = \frac{x_4}{x_5} = \frac{x_5}{x_6},$$

defining six linearly independent quadratic relations. This system defines a rational ruled surface of order 4, common to all the six quadrics, which are therefore not sufficient to define the curve.*

On the other hand, it is not difficult to discuss these curves directly from their equations in the plane. The general form is

$$f_3(x, y) z^3 + f_4(x, y) z^2 + \psi_4(x, y) xz + \psi_3(x, y) x^2 y = 0.$$

If $\psi_3(x, y) = f_3(y, x)$ and $\psi_4(x, y) = f_4(y, x)$, c_6 is invariant under $\rho x' = yz$, $\rho y' = zx$, $\rho z' = xy$. If in addition $f_4 \equiv 0$, c_6 has the cyclic perspectivity

$$\sigma x' = x,\quad \sigma y' = y,\quad \sigma z' = \omega z,\quad \omega^3 = 1.$$

The latter can exist alone if $f_4 \equiv 0$, $\psi_4 \equiv 0$.

In particular, the curve

$$x^2 y (ax^3 + by^3) + z^3 (bx^3 + ay^3) = 0$$

has the quadratic inversion and

$$\rho x' = \theta^4 x,\quad \rho y' = \theta y,\quad \rho z' = z,\quad \theta^9 = 1,$$

defining the dihedral G_{18}. The forms having a G_4 generated by a harmonic homology about x or y and the inversion can be immediately written down.

The curve

$$x^3 z^3 + (ax^4 + by^4) z^2 + (cx^4 + dy^4) xz + kx^2 y^4 = 0$$

has the cyclic G_4 defined by $\begin{pmatrix} x & y & z \\ x & iy & z \end{pmatrix}$. In particular, if $c = b$, $d = a$, $k = 1$, it also admits the quadric inversion, thus defining a dihedral G_8. The point (0, 0, 1) has $x = 0$ for triple tangent; at the double point (0, 1, 0) each tangent has five-point contact. The line $y = 0$ meets c_6 in three other points, at each of which the tangent has four-point contact and passes through the double point. The curve has 32 other points of inflexion, arranged on eight lines passing through the double point.

Of the two forms having four coincident double points, that with a simple branch passing through a tacnode may have at most a single harmonic homology, as

$$axy^2 z^3 + by^4 z^2 + y^2 \phi_2(x^2, y^2) + xy^2 z (cx^2 + dy^2) + cx^5 z + fx^2 y^2 z^2 = 0.$$

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*Kraus, *l. c.*; Snyder, "On Birational Transformations of Curves of High Genus," JOURNAL, Vol. XXX 1908), p. 10.

That with a simple branch passing through a cusp of the second kind,

$$zy(x-ay^2)^2+bx^2y^2z^2+x^2yzf_2(x,y)+cx^3yz^2+dx^4z^2+x^2\phi_4(x,y)=0,$$

has no invariant transformations.

§ 4 (d). *The Non-singular Quintic.*

14. It has been shown* that if a curve of genus 6 has a g_5^2 it could not be reduced to a sextic. The non-singular curves have at most only linear transformations into themselves. The forms of the possible linear groups to which c_5 can belong have already been determined.†

The adjoint curves are made up of all the conics of the plane. If we write

$$\rho x_1=x^2,\quad \rho x_2=xy,\quad \rho x_3=y^2,\quad \rho x_4=xz,\quad \rho x_5=yz,\quad \rho x_6=z^2,$$

then

$$\frac{x_1}{x_2}=\frac{x_2}{x_3}=\frac{x_4}{x_5},\quad \frac{x_1}{x_4}=\frac{x_4}{x_6}=\frac{x_2}{x_5},\quad \frac{x_2}{x_4}=\frac{x_3}{x_5}=\frac{x_5}{x_6},$$

or

$$x_1x_3=x_2^2,\quad x_1x_5=x_2x_4,\quad x_2x_5=x_3x_4,\quad x_1x_6=x_4^2,\quad x_4x_5=x_2x_6,\quad x_3x_6=x_5^2.$$

Hence, here too, the six quadratic relations are independent of the quintic curve. This is the only case thus far discovered of a curve not having a g_3^1 which is not defined by the quadratic relations among the adjoint curves. The six quadrics have a surface in common, but not a ruled surface. It is the Veronese surface of order 4. It can be projected from $(0, 0, 0, 0, 0, 1)$ into $x_6=0$ as the rational ruled surface of order 3,

$$\frac{x_1}{x_2}=\frac{x_2}{x_3}=\frac{x_4}{x_5},$$

and therefore, from the preceding section, it also follows that *if the normal curve be projected from a bisecant, it projects into a conic, counted four times.* We have three interesting projections into R_3. If the surface be projected from any bisecant, the result is a quadric surface. If the line have but one point in common with the surface in R_5, the result is a ruled cubic of the first kind, as

$$x_3x_4^2=x_1x_5^2.$$

Finally, by projecting from a line not having any point on the surface, we obtain, for example,

$$\sqrt{x_1+x_3+x_6+2(x_2+x_4+x_5)}=\sqrt{x_1}+\sqrt{x_3}+\sqrt{x_6},$$

a Steiner surface.‡

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*Snyder, *l. c.*

† Snyder, "Plane Quintic Curves Which Possess a Group of Linear Transformations," JOURNAL, Vol. XXX (1908), p. 1. The most interesting type is $x^5+y^5+z^5=0$, which is invariant under a group of order 150.

‡ An excellent discussion of the Veronese surface is given by Bertini, *Introduzione alla Geometria Proiettiva degli Iperspazi*, Pisa, 1907. See Chapter XV.

§ 5 (e). *Hyperelliptic Curves.*

15. The canonical form of a hyperelliptic curve of genus 6 is

$$y^2 z^{12} = f_{14}(x, z).$$

The characteristic G_2 is the homology

$$\begin{pmatrix} x & y & z \\ x & -y & z \end{pmatrix} = H.$$

If $f_{14}(x, z) = \phi_7(x^2, z^2)$, we have the non-cyclic G_4. If $f_{14}(x, z) = f_{14}(z, x)$, another G_4, defined by H and

$$K \equiv \begin{pmatrix} x & y & z \\ x^6 z & y z^6 & x^7 \end{pmatrix}.$$

The dihedral G_8 arises when $\phi_7(x^2, z^2) = \phi_7(z^2, x^2)$.

By inversion, the equation of the curve may be written

$$y^2 z^{11} = f_{13}(x, z).$$

If $f_{13}(x, z) = x\phi_6(x^2, z^2)$, we have the cyclic

$$G_4 \equiv \begin{pmatrix} x & y & z \\ -x & iy & z \end{pmatrix}.$$

If $\phi_6(x^2, z^2) = \phi_6(z^2, x^2)$, the curve also admits k, making a dihedral G_8.

$y^2 z^{11} = x f_4(x^3, z^3)$ has $\begin{pmatrix} x & y & z \\ \theta^2 x & \theta y & z \end{pmatrix}$, $\theta^6 = 1$; if $f_4(x^3, z^3) = f_4(z^3, x^3)$, the dihedral G_{12}; $y^2 z^{11} = x f_3(x^4, z^4)$, the cyclic $G_8 \equiv \begin{pmatrix} x & y & z \\ ix & \sqrt{i}y & z \end{pmatrix}$; and if f_3 is symmetric, the dihedral G_{16}. In particular, $y^2 z^{11} = x(x^4 + z^4)(x^8 - 14x^4 z^4 + z^8)$ has a G_{48}, formed by H and the octahedron group.

$y^2 z^{11} = x(x^{12} + z^{12})$ has the dihedral G_{48}.

$y^2 z^{11} = x f_2(x^6, z^6)$ has the cyclic $G_{12} \equiv \begin{pmatrix} x & y & z \\ \theta x & \sqrt{\theta}y & z \end{pmatrix}$ and, if f_2 is symmetric, the dihedral G_{24}; $y^2 z^{11} = x^{13} + z^{13}$, the cyclic G_{26}. This is the only operation of period as high as 26 that any curve of genus 6 can have.

$y^2 z^{12} = x^{14} + z^{14}$ has the dihedral G_{28}, and H, making a group of order 56.*

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* A. Wiman, "Ueber die hyperelliptischen Curven und diejenigen vom Geschlecht $p=3$, welche eindeutige Transformationen in sich zulassen," *Bihang t. k. Svenska Vetenskab Akad. Handlingar*, Band XXI (1895).

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